11 Determinants, the Cramer rule

(Applied Mathematics — FAPPZ)

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Contents

1 Determinant

1.1 Definition of determinant

Definition 1. Permutation π of the set $\{1, 2, \dots, n\}$ is a one to one mapping of this set into itself (that is $\pi : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$, where $\mathcal{D}(\pi) = \mathcal{H}(\pi) = \{1, 2, \dots, n\}$ and the mapping π is one to one).

The permutation π can be written as:

$$\pi = \begin{pmatrix} 1 & 2 & \dots & n \\ \pi(1) & \pi(2) & \dots & \pi(n) \end{pmatrix}.$$

Definition 2. An *inversion* in permutation π is a pair of entries $i, j \in \{1, 2, \dots, n\}$ such that $i < j$ and $\pi(i) > \pi(j)$.

An *even* (or *odd*) permutation is a permutation which contains even (or odd) number of inversions.

By the symbol Π_n we denote the set of all permutations of the set $\{1, 2, \dots, n\}$. For $\pi \in \Pi_n$ let k_π be the number *of all inversions* of π.

The *determinant* is a special number associated to any square matrix.

Definition 3. If $\boldsymbol{A}$ is a square matrix of degree n, we define its *determinant* by the following way

$$\det \boldsymbol{A} = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = \sum_{\pi \in \Pi n} (-1)^{k_\pi} a_{1\pi(1)} a_{2\pi(2)} \dots a_{n\pi(n)}.$$

1.2 Determinant of degree 1, 2, or 3

1. For $n = 1$ we have $\det(a) = a$.
2. For $n = 2$,
$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$
3. For $n = 3$ we use the *Sarrus rule*:
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}.$$

ATTENTION!!!
No analogue of the Sarrus rule holds for $n \geq 4$.

1.3 Cofactor

Definition 4. If $i, j \in \{1, 2, \dots, n\}$, we denote by $\boldsymbol{M}_{ij}$ the matrix of degree $n-1$ obtained by deleting ith row and jth column of the square matrix $\boldsymbol{A}$. The number $A_{ij} = (-1)^{i+j} \det \boldsymbol{M}_{ij}$ is called the *cofactor* associated with the (i, j)-position of the matrix $\boldsymbol{A}$.

Example 5. For $\boldsymbol{A} = \begin{pmatrix} 3 & 2 & 4 \\ -1 & 3 & 1 \\ 5 & 1 & 6 \end{pmatrix}$, $i = j = 2$, $\boldsymbol{M}_{22} = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$, and so

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} = 1 \cdot (18 - 20) = -2.$$

1.4 Expanding a determinant about a row or a column

Expanding about a row
Let $i \in \{1, 2, \dots, n\}$ then

$$\det \boldsymbol{A} = a_{i1}A_{i1} + a_{i2}A_{i2} + \dots + a_{in}A_{in}.$$

Expanding about a column
Let $j \in \{1, 2, \dots, n\}$ then

$$\det \boldsymbol{A} = a_{1j}A_{1j} + a_{2j}A_{2j} + \dots + a_{nj}A_{nj}.$$

1.5 Calculating a determinant using row transformations

Definition 6. A square matrix $\boldsymbol{A} = (a_{ij})_{i,j=1}^n$ is called an upper triangle matrix, if $a_{ij} = 0$ for all indices $i, j \in \{1, \ldots, n\}$, $i > j$, and it is called a lower triangle matrix, if $a_{ij} = 0$ for all indices $i, j \in \{1, \ldots, n\}$, $i < j$.

Theorem 7. *If $\boldsymbol{A} = (a_{ij})_{i,j=1}^n$ is an upper triangle or a lower triangle matrix, then*

$$\det \boldsymbol{A} = a_{11}\, a_{22} \cdots a_{nn}.$$

Theorem 8. *Let $\boldsymbol{B}$ be a square matrix which results from the matrix $\boldsymbol{A}$ of the same degree by*

1. *interchanging two rows, then $\det \boldsymbol{B} = -\det \boldsymbol{A}$;*
2. *multiplying one row with a number $r \in \mathbb{R}$, then $\det \boldsymbol{B} = r \det \boldsymbol{A}$;*
3. *adding a multiple of one row to another row, then $\det \boldsymbol{B} = \det \boldsymbol{A}$.*

Remark 9. The same rules hold with columns instead of rows.

2 Calculating a matrix inverse by determinants

2.1 A general formula

Theorem 10. *A square matrix $\boldsymbol{A}$ of degree n is regular if and only if $\det \boldsymbol{A} \neq 0$.*

Theorem 11. *Let $\boldsymbol{A}$ be a square regular matrix of degree n. Then it has the inverse $\boldsymbol{A}^{-1}$ and*

$$\boldsymbol{A}^{-1} = \frac{1}{\det \boldsymbol{A}} \cdot (A_{ij})^T = \frac{1}{\det \boldsymbol{A}} \begin{pmatrix} A_{11} & A_{21} & \ldots & A_{n1} \\ A_{12} & A_{22} & \ldots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \ldots & A_{nn} \end{pmatrix},$$

where the numbers A_{ij} are cofactors to the positions (i, j), $i, j \in \{1, \ldots, n\}$, of the matrix $\boldsymbol{A}$.

2.2 A formula for a matrix of degree 2

If

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

then

$$\boldsymbol{A}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \cdot \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$$

3 The Cramer rule

Consider a system of n linear equations with n variables

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + \ldots + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + \ldots + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + \ldots + & a_{nn}x_n & = & b_n \end{array} \tag{1}$$

with a *regular* matrix $\boldsymbol{A}$. Let $\boldsymbol{A}_{(j)}$ be the matrix obtained replacing jth column, $j \in \{1, \ldots, n\}$, of $\boldsymbol{A}$ by the column $\mathbf{b}$ of right hand sides, that is,

$$\boldsymbol{A}_{(j)} = \begin{pmatrix} a_{11} & \ldots & b_1 & \ldots & a_{1n} \\ a_{21} & \ldots & b_2 & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \ldots & b_n & \ldots & a_{nn} \end{pmatrix}.$$

Then the solution $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ of (1) satisfies

$$x_j = \frac{\det \boldsymbol{A}_{(j)}}{\det \boldsymbol{A}}, \quad j = 1, 2, \ldots, n.$$