2 Limit of a function, continuity of a function, derivatives

(Applied Mathematics — FAPPZ)

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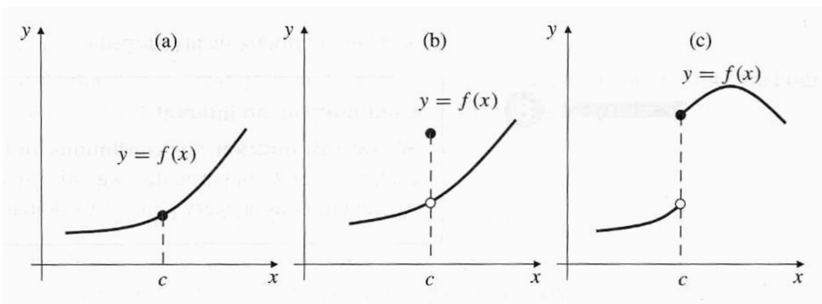
(Updated on October 2, 2011)

Contents

1 Continuity and a limit of a function



- Function f on picture (a) has a limit at c and is continuous at c.
- Function f on picture (b) has a limit at c but it is not continuous at c.
- Function f on picture has no limit at c and so it is not continuous at c. It is right continuous at c.

1.1 Definition of limit at a point

Definition 1 (Neighbourhood)**.** Let $a \in \mathbb{R}$, $\delta > 0$. The set

- $U_\delta(a) := (a - \delta, a + \delta)$ is *δ-neighbourhood* of point a;
- $P_\delta(a) := (a - \delta, a) \cup (a, a + \delta)$ *reduced δ-neighbourhood* of point a.

If $a \in \{-\infty, +\infty\}$ we define

- $U_\delta(+\infty) = P_\delta(+\infty) := (1/\delta, +\infty)$;
- $U_\delta(-\infty) = P_\delta(-\infty) := (-\infty, -1/\delta)$.

Definition 2 (Limit)**.** Let $c, A \in \mathbb{R} \cup \{-\infty, +\infty\}$. Function f has at c limit A, we write $\lim_{x \to c} f(x) = A$, if for any $U_\varepsilon(A)$ there exists $P_\delta(c) \subset \mathcal{D}(f)$, such that for each $x \in P_\delta(c)$ we have $f(x) \in U_\varepsilon(A)$.

Remarks 3.

- If $c \in \{-\infty, +\infty\}$, we say that the limit is at (plus or minus) infinity.
- If $A \in \{-\infty, +\infty\}$, we say that the limit is infinite.
- If, for $c \in \mathbb{R}$, we replace $P_\delta(c)$ by the left neighbourhood $P_\delta^-(c) := (c-\delta, c)$, we call the corresponding limit as left limit and write $\lim_{x \to c^-}$.
- Similarly, if, for $c \in \mathbb{R}$, we replace $P_\delta(c)$ by the right neighbourhood $P_\delta^+(c) := (c, c+\delta)$, we are talking about the right limit and write $\lim_{x \to c^+}$.

1.2 Continuity of a function at a point (local continuity)

Definition 4. Function f is continuous at a point c if there exists $\lim_{x \to c} f(x)$ and $\lim_{x \to c} f(x) = f(c)$.

Remark 5. Replacing the limit in the definition by left limit, or by right limit, we say that the function is left continuous, or right continuous, respectively.

Theorem 6 (Continuity of elementary functions)**.** *Let f be an elementary function with the domain $\mathcal{D}(f)$ and let $c \in \mathcal{D}(f)$. Then*

- *If $U_\delta(c) \subset \mathcal{D}(f)$ for some $\delta > 0$ (that is, the point c is an interior point of the domain of f), then f is continuous at c.*
- *If $U_\delta^-(c) \subset \mathcal{D}(f)$ for some $\delta > 0$, then f is left continuous at c.*
- *If $U_\delta^+(c) \subset \mathcal{D}(f)$ for some $\delta > 0$, then f is right continuous at c.*

2 Derivative at a point

2.1 Definition of the first derivative

Definition 7 (Derivative at a point)**.** Let f be a function and let a be a real number. Then *the first derivative* $f'(a)$ of the function f at the point a is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

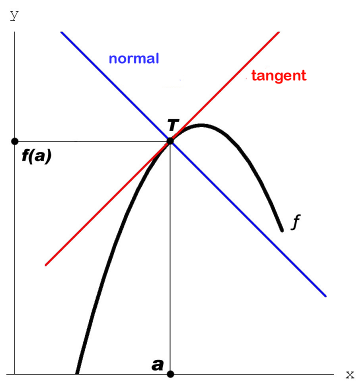
The derivative might be plus or minus infinity. Then we say that the derivative is infinite. If the limit does not exist we say that the function does not have derivative at a.

Instead of $f'(a)$ we also use the notation $\frac{df}{dx}(a)$, $\frac{d}{dx}f(a)$.

Remark 8.
1. If we replace the limit in Definition 7 by right limit, or by left limit, we call the derivative right derivative, or left derivative, respectively.
2. The first derivative $f'(a)$ may exist only if the point a is an interior point of the domain $\mathcal{D}(f)$ of f, i. e. $U_\delta(a) \subset \mathcal{D}(f)$ for some $\delta > 0$.
3. Left derivative (or right derivative) at a may exist only if $U_\delta^-(a) \subset \mathcal{D}(f)$ (or $U_\delta^+(a) \subset \mathcal{D}(f)$) for some $\delta > 0$.

3 Derivative and its geometric meaning

3.1 Tangent line and normal



Definition 9. Let the derivative $f'(a)$ at point a exists. The line t,

$$t : y - f(a) = f'(a)(x - a)$$

is called the *tangent line* to the graph of f at the point $T = [a, f(a)]$.

Normal is the line perpendicular to the tangent line t passing through the point T.

Definition 10. Let the derivative $f'(a)$ at point a exists. The line n,

$$n : \ x = a \quad \text{for } f'(a) = 0,$$
$$n : \ y - f(a) = \frac{-1}{f'(a)}(x - a) \quad \text{for } f'(a) \neq 0,$$

we call the *normal* to the graph of f at the point $T = [a, f(a)]$.