3 Monotony, local and global extreme values of a function

(Applied Mathematics — FAPPZ)

Petr Gurka

(Updated on October 24, 2017)

Contents

1 Global continuity of a function

1.1 Continuity on an interval

Definition 1. Let $\mathcal{J} \subset \mathcal{D}(f)$ be an interval of arbitrary type.

- If $\mathcal{J}$ is an *open* interval, then a function f is continuous on $\mathcal{J}$ if it is continuous at each of its points.
- If $\mathcal{J}$ is *half-closed* or *closed* interval, then function f is continuous on $\mathcal{J}$ if it is continuous at each of its interior points and one sided continuous at corresponding endpoint(s). (That is, if this interval contains its left (right) endpoint then the function is right (left) continuous at this endpoint.)

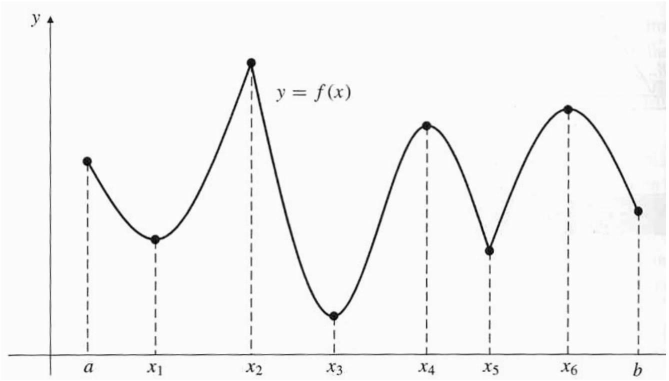
Theorem 2 (Continuity of elementary functions)**.** *Let f be an elementary function and let $\mathcal{J}$ be an interval (of arbitrary type) which is a subset of the domain of function f. Then f is continuous on $\mathcal{J}$.*

1.2 Properties of functions continuous on an interval

Remark 3.
- If f is continuous on an interval $\mathcal{I}$, then the set
$$f(\mathcal{I}) := \{y = f(x) \,|\, x \in \mathcal{I}\}$$
is also an interval.
- If f is continuous and *one to one* on an interval $\mathcal{I}$, then both intervals $\mathcal{I}$ and $f(\mathcal{I})$ are of the same type and their endpoints correspond each other.

2 Monotone functions

2.1 Function monotone on an interval



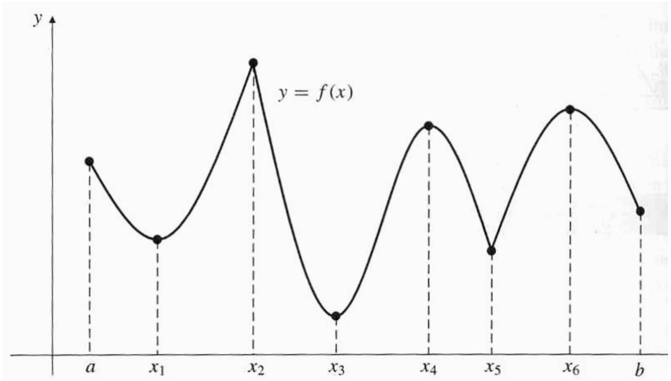
Domain: $\mathcal{D}(f) = \langle a, b\rangle$
Function f is *increasing* on intervals: $\langle x_1, x_2\rangle$, $\langle x_3, x_4\rangle$, $\langle x_5, x_6\rangle$.
Function f is *decreasing* on intervals: $\langle a, x_1\rangle$, $\langle x_2, x_3\rangle$, $\langle x_4, x_5\rangle$, $\langle x_6, b\rangle$.

Definition 4. Let f be a function with its domain $\mathcal{D}(f)$ and let $\mathcal{J} \subset \mathcal{D}(f)$ is a nonempty interval (of arbitrary type). We say that f is

- *increasing* on $\mathcal{J}$, if for all $x_1, x_2 \in \mathcal{J}$ such that $x_1 < x_2$, $f(x_1) < f(x_2)$;
- *decreasing* on $\mathcal{J}$, if for all $x_1, x_2 \in \mathcal{J}$ such that $x_1 < x_2$, $f(x_1) > f(x_2)$.

Function which is either *increasing* or *decreasing* on interval $\mathcal{J}$, is called *(strictly) monotone* on $\mathcal{J}$.

2.2 Local extreme values



Domain: $\mathcal{D}(f) = \langle a, b \rangle$
Local *minimum* values at points: x_1, x_3, x_5.
Local *maximum* values at points: x_2, x_4, x_6.
No local extreme values at endpoints a, b.

Definition 5. We say that a function f has at point $a \in \mathcal{D}(f)$

- *local minimum value* if there exists $\delta > 0$ such that for all $x \in (a-\delta, a+\delta)$, $f(x) \geq f(a)$;
- *local maximum value* if there exists $\delta > 0$ such that for all $x \in (a-\delta, a+\delta)$, $f(x) \leq f(a)$.

IMPORTANT.
Necessarily $(a - \delta, a + \delta) \subset \mathcal{D}(f)$, that is, local extremal values might be only at interior points of the domain.

2.3 Necessary condition for the existence of local extreme values

Theorem 6. *If a function f at point $a \in \mathcal{D}(f)$ attains its local extreme value and it has a derivative $f'(a)$ at a, then $f'(a) = 0$.*

Remark 7 (Advantage). Easy calculation: we can find "suspicious" points by solving the equation $f'(x) = 0$.

Remark 8 (Disadvantages).

- We find also other points of no extreme values.
- We do not find points of extreme values in which function f does not have a derivative.

2.4 Monotony by using the first derivative

Theorem 9. *Let a function f has its first derivative $f'(x)$ at all points of the open interval $(a, b) \subset \mathcal{D}(f)$. Then*

- *if $f'(x) > 0$ at each $x \in (a, b)$, then f is increasing on interval (a, b);*

- *if $f'(x) < 0$ at each $x \in (a, b)$, then f is decreasing on interval (a, b);*
- *if $f'(x) = 0$ at each $x \in (a, b)$, je funkce f is constant on interval (a, b).*

Moreover, if f is (one sided) continuous at one of the endpoints a, b, then the property is valid on the corresponding (closed or half-closed) interval containing this endpoint(s).

Theorem 10. *Let $a < c < b$ and f is continuous at c. Then*

- *if f is increasing on intervals (a, c) and (c, b), it is also increasing on interval (a, b);*
- *if f is decreasing on intervals (a, c) and (c, b), it is also decreasing on interval (a, b).*

Theorem 11 (Using monotony to find local extreme values)**.** *Let $a < c < b$ and f is continuous at c. Then*

- *if the function f is increasing on (a, c) and decreasing on (c, b), then it has at c local minimum value;*
- *if the function f is decreasing on (a, c) and increasing on (c, b), then it has at c a local minimal value.*

3 Global extreme values

3.1 Definition of a global extreme value

Definition 12. Let f be a function with a domain $\mathcal{D}(f)$ and $\emptyset \neq M \subset \mathcal{D}(f)$. We say that f has at $a \in M$ *global maximum value* (or *global minimum value*) *on the set M* if, for each $x \in M$,

$$f(x) \leq f(a) \quad (\text{or } f(x) \geq f(a)).$$

Global minimum value or global maximum value we call *global extreme values*.

3.2 Existence of global extreme values

Theorem 13 (Weierstrass)**.** *Let function f is continuous on a closed interval $\langle a, b\rangle \subset \mathcal{D}(f)$. Then there exist points $c_1, c_2 \in \langle a, b\rangle$ such that function f has at c_1 global maximum value and at c_2 global minimum value on $\langle a, b\rangle$.*

Remark 14. A continuous function on closed interval has global extreme values either at points of local extreme values or at the endpoints.