4 Convexity, concavity, inflection, sketching a graph

(Applied Mathematics — FAPPZ)

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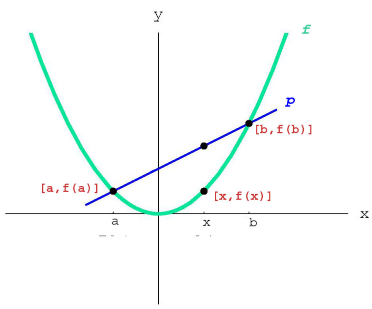
Contents

1 Convexity and concavity

1.1 Convex and concave functions

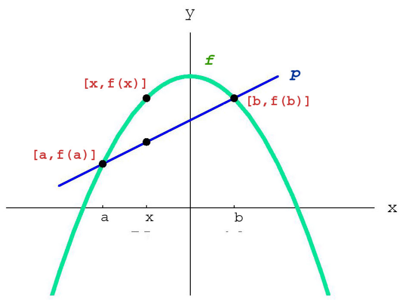
Definition 1. We call a function f *convex* on interval $\mathcal{J}$ if, for all $a, b \in \mathcal{J}$, $a < b$, and each $x \in (a, b)$,

$$f(x) < \frac{f(b) - f(a)}{b - a}(x - a) + f(a).$$



Definition 2. We call a fuction f *concave* on interval $\mathcal{J}$ if, for all $a, b \in \mathcal{J}$, $a < b$, and each $x \in (a, b)$,

$$f(x) > \frac{f(b) - f(a)}{b - a}(x - a) + f(a).$$



1.2 Using the second derivative to determine convexity and concavity

Theorem 3. *Let f has its second derivative $f''(x)$ on interval $(a,b) \subset \mathcal{D}(f)$. Then one of the following statements holds:*

- *if $f''(x) > 0$ at each $x \in (a,b)$, then f is convex on (a,b);*
- *if $f''(x) < 0$ at eachx $\in (a,b)$, then f is concave on (a,b).*

If the function is (right or left) continuous at an endpoint a, b, then convexity or concavity holds on the interval containing this endpoint.

Attention!
In general, a function which is convex (or concave) on intervals (a,c) and (c,b) need not be convex (or concave) on (a,b), even if it is continuous at c.

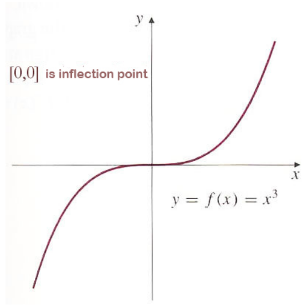
Theorem 4. *Let $a < c < b, \quad f''(c) = 0$. Then*

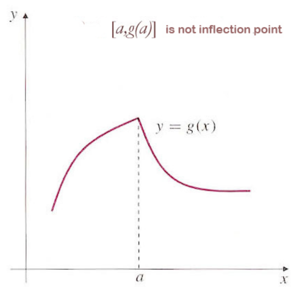
- *if $f'' > 0$ on (a,c) and (c,b), then f is convex on (a,b);*
- *if $f'' < 0$ on (a,c) and (c,b), then f is convex on (a,b).*

1.3 Inflection point of a graph

Definition 5. We say that $A = [a, f(a)]$ is an *inflection point* of the graph of f, if

- point a is an interior point of the domain of f, (i. e. for some $\delta > 0$, $(a-\delta, a+\delta) \subset \mathcal{D}(f)$);
- function f has at a its first derivative $f'(a)$;
- on one of the intervals $(a-\delta, a\rangle$, $\langle a, a+\delta)$ the function f is convex and on the other of them it is concave.





2 Sketching the graph of a function

We should investigate:

- domain of the function,
- elementary properties of the function (e. g. we determine points in which the graph intersects the coordinate axes, the symmetry of the graph etc.)
- limits at endpoints of the domain of the function (we can also find asymptotes),
- intervals of increase and decrease, local extreme values (using the first derivative),
- convexity, concavity and inflection points (using the second derivative),
- coordinates of important points on the graph (local extreme values, inflection poits etc.),
- sketching the graph.