6 Differential equations

(Applied Mathematics — FAPPZ)

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1 Differential equations

1.1 What we mean by differential equation

By differential equation we mean an equation for an unknown function containing its derivatives (it may also contain the unknown function). The unknown function (of variable x) we denote by y and its derivatives by y', y'' etc. We shall consider only differential equations of the first order, that is, these equations contain only the first derivative y' of the unknown function y.

Example 1. Basic example of a differential equation of the first order is

$$y' = f(x).$$

Solution of this equation is an arbitrary antiderivative $y = F(x)$ to f on a suitable interval $\mathcal{J}$.

2 Separable equations

2.1 Definitions

Definition 2 (Separable equation)**.** Differential equation of the type

$$y' = f(x)\, g(y) \tag{1}$$

we call a *separable equation.*

Remark 3. By separable equations we also consider the equations

$$u(y)\, y' = v(x), \quad y' = \frac{v(x)}{u(y)}.$$

Definition 4 (Solution)**.** We say that a function φ defined on an open interval (a, b) is a *solution of differential equation* (1) *on interval* (a, b), if, at each point $x \in (a, b)$, it has the first derivative $\varphi'(x)$ and

$$\varphi'(x) = f(x)\, g\big(\varphi(x)\big).$$

(We can define solution also on a closed or half-closed interval. Then we consider at the endpoint(s) only one-sided derivative(s).)

How to solve a separable equation (1)

1. We write the derivative y' in the form $y' = \frac{\mathrm{d}y}{\mathrm{d}x}$, the corresponding equation we formally multiply by the symbol $\mathrm{d}x$ and divide by $g(y)$. Then we add to both sides a symbol $\int$. That is,
$\dfrac{\mathrm{d}y}{\mathrm{d}x} = f(x)\, g(y) \Rightarrow \dfrac{\mathrm{d}y}{g(y)} = f(x)\,\mathrm{d}x \Rightarrow \displaystyle\int \frac{\mathrm{d}y}{g(y)} = \int f(x)\,\mathrm{d}x.$

2. We compute integrals on both sides to obtain $G(y) = F(x) + C$, from which, we express y by x, that is, $\varphi : y = G^{-1}\big(F(x) + C\big)$.

3. The function φ just obtained we put into the equation (1) and find on which interval it is a solution of the equation (1).

4. If the function g has a zero point x_0, i.e., $g(x_0) = 0$, then the constant function $\varphi_s : y = x_0$ is also a solution of the equation (1) on any interval $(a, b) \subset \mathcal{D}(f)$

2.2 Initial problem for differential equation

Definition 5 (Initial problem)**.** By *initial problem* for differential equation (1) we mean

$$\begin{cases} y' & = & f(x)\, g(y), \\ y(x_0) & = & y_0. \end{cases}$$

Its solution φ, is a solution satisfying the *initial condition*

$$\varphi(x_0) = y_0,$$

(necessarily $x_0 \in \mathcal{D}(f)$ and $y_0 \in \mathcal{D}(g)$, that is, $f(x_0)$ and $g(y_0)$ make sense). In other words, the graph of the solution φ passes through the point $[x_0, y_0]$.