7 Definite integral

(Applied Mathematics — FAPPZ)

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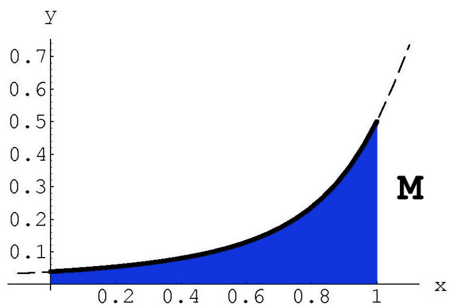
Contents

1 Definite integral

Consider a nonnegative continuous function f on a closed interval $\langle 0,1\rangle$. We want to evaluate the area of the region M in the plane bounded from above by the graph of the function f (that is, by the curve $y = f(x)$, $x \in \langle 0,1\rangle$), from below by the x-axis, from the left by the y-axis and from the right by the line $x = 1$.



Idea how to compute the area of M.
Replace the *region* M by a *"simpler"* region whose area we can compute precisely and such that its area is not too much different from the area $P(M)$ of M.

Split the interval $\langle 0,1\rangle$ into subintervals with endpoints $x_0, x_1, \ldots, x_n \in \langle 0,1\rangle$, $n \in \mathbb{N}$, setting $x_0 = 0$ and $x_n = 1$, that is,

$$0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1,$$

and consider the region Q, which is a union of rectangles with the basis $\langle x_{i-1}, x_i\rangle$ and with heights equal to $f(c_i)$, where c_i are suitably chosen points from corresponding intervals $\langle x_{i-1}, x_i\rangle$, $i = 0, 1, \ldots, n$. We can compute the *area* $P(Q)$ of the region Q *precisely*,

$$P(Q) = f(c_1)\,(x_1 - x_0) + f(c_2)\,(x_2 - x_1) + \cdots + f(c_n)\,(x_n - x_{n-1}). \qquad (1)$$

If we choose c_i so that

$$f(c_i) = \min_{x \in \langle x_{i-1}, x_i\rangle} f(x), \quad i \in \{1, \ldots, n\},$$

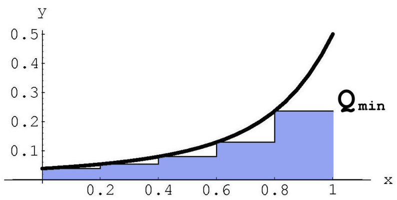
then the corresponding region $Q_{\min}$ is a *subset* of M.

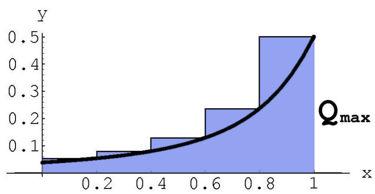
If we choose c_i so that

$$f(c_i) = \max_{x \in \langle x_{i-1}, x_i\rangle} f(x), \quad i \in \{1, \ldots, n\},$$

then M is a subset of $Q_{\max}$. Consequently, areas of these regions satisfy the following inequalities

$$P\big(Q_{\min}\big) \le P\big(M\big) \le P\big(Q_{\max}\big)$$





It is easy to see that the difference between the precise area $P\big(M\big)$ and $P(Q)$ from (1) with arbitrary chosen c_i, $i = 0, 1, \ldots, n$, is *at most* $P\big(Q_{max}\big) - P\big(Q_{min}\big)$.

1.1 Riemann definite integral

Definition 1. Given a closed interval $\langle \alpha, \beta\rangle$. The set of points

$$D = \{x_0, x_1, \ldots, x_n\} \subset \langle \alpha, \beta\rangle$$

satisfying

$$\alpha = x_0 < x_1 < \cdots < x_{n-1} < x_n = \beta$$

we call a *partition of the interval* $\langle \alpha, \beta\rangle$. *Norm* of the partition D, denoted by $\nu(D)$, is defined as

$$\nu(D) = \max_{i \in \{0,1,\ldots,n\}} (x_i - x_{i-1}).$$

For a given partition D let $B = \{c_1, \dots, c_n\}$ is a set of points from the interval $\langle \alpha, \beta \rangle$ such that $c_i \in \langle x_{i-1}, x_i \rangle$, $i = 1, 2, \dots, n$. Denote

$$s(f, D, B) = \sum_{i=1}^{n} f(c_i)\,(x_i - x_{i-1})$$

the *integral sun* associated to the partition D and the choice of points B.

Definition 2 (Riemann definite integral)**.** Let f be a bounded function on the closed interval $\langle \alpha, \beta \rangle$. Let there exist a number R such that, for any $\varepsilon > 0$, there is $\delta > 0$ so that, for an arbitrary partition D of the interval $\langle \alpha, \beta \rangle$ with norm $\nu(D) < \delta$, and, for arbitrary choice $c_i \in \langle x_{i-1}, x_i \rangle$, $i = 1, 2, \dots, n$,

$$|R - s(f, D, B)| < \varepsilon.$$

Then we say, that the function f is *Riemann integrable* on the interval $\langle \alpha, \beta \rangle$. The number R we call the Riemann definite integral of f on $\langle \alpha, \beta \rangle$. We write

$$\int_{\alpha}^{\beta} f(x)\,dx = R.$$

If no number R of these properties *exists* we say that f *is not* Riemann integrable on $\langle \alpha, \beta \rangle$.

Roughly speaking,

$$\int_{\alpha}^{\beta} f(x)\,dx = \lim_{\nu(D) \to 0}\ s(f, D, B),$$

where the limit is *independent* of the partition D and of the choice of points B.

1.2 Fundamental theorem of calculus

Theorem 3 (Sufficient condition for Riemann integrability)**.** *Let the function f is continuous on a closed interval $\langle \alpha, \beta \rangle$. Then it is Riemann integrable on the interval $\langle \alpha, \beta \rangle$.*

Theorem 4 (Fundamental theorem of calculus)**.** *Let the function f is Riemann integrable on the closed interval $\langle \alpha, \beta \rangle$ and let f has antiderivative F on $\langle \alpha, \beta \rangle$. Then*

$$\int_{\alpha}^{\beta} f(x)\,dx = F(\beta) - F(\alpha). \tag{2}$$

(The assumptions of the theorem are satisfied if f is continuous on $\langle \alpha, \beta \rangle$.)

Remark 5. We can define a definite integral by (2). This integral is called *Newton definite integral.*

Definition 6. Let $\alpha, \beta \in \mathbb{R}$, $\alpha > \beta$. We define (cf. (2))

$$\int\limits_{\alpha}^{\beta} f(x)\ \mathrm{d}x = -\int\limits_{\beta}^{\alpha} f(x)\ \mathrm{d}x, \qquad \int\limits_{\alpha}^{\alpha} f(x)\,dx = 0.$$

Theorem 7. *Let $\alpha, \beta, \gamma \in \mathbb{R}$, $\alpha < \gamma < \beta$ and $c \in \mathbb{R}$. Then*

$$\int_{\alpha}^{\beta} c\, f(x)\, dx = c \int_{\alpha}^{\beta} f(x)\, dx,$$

$$\int_{\alpha}^{\beta} \big(f(x) \pm g(x)\big)\, dx = \int_{\alpha}^{\beta} f(x)\, dx \pm \int_{\alpha}^{\beta} g(x)\, dx,$$

$$\int_{\alpha}^{\beta} f(x)\, dx = \int_{\alpha}^{\gamma} f(x)\, dx + \int_{\gamma}^{\beta} f(x)\, dx,$$

whenever definite integrals on both sides exist.

1.3 Method of integration by parts for definite integrals

Notation 8. For the number $F(\beta) - F(\alpha)$ on the right-hand side of (2) we use the *evaluation symbol*

$$F(\beta) - F(\alpha) = \big[F(x)\big]_{\alpha}^{\beta}.$$

Theorem 9 (Integration by parts)**.** *Let functions u, v have continuous derivatives u', v' on an open interval $\langle \alpha, \beta \rangle$. Then*

$$\int_{\alpha}^{\beta} u(x)\, v'(x)\, dx = \big[u(x)\, v(x)\big]_{\alpha}^{\beta} - \int_{\alpha}^{\beta} u'(x)\, v(x)\, dx.$$

1.4 Method of substitution for definite integrals

Theorem 10 (Substitution in a definite integral)**.** *Let f be a continuous function on $\langle A, B \rangle$. Let g be a function which has continuous first derivative g' on interval $\langle \alpha, \beta \rangle$ and let*

$$g(x) \in \langle A, B \rangle \quad \textit{for all } x \in \langle \alpha, \beta \rangle.$$

Then

$$\int_{\alpha}^{\beta} f\big(g(x)\big)\, g'(x)\, dx = \int_{g(\alpha)}^{g(\beta)} f(t)\, dt.$$

Attention!
After a change of variables in a definite integral and transformation of limits of integration we do not return to the original variable.

2 Applications of definite integrals

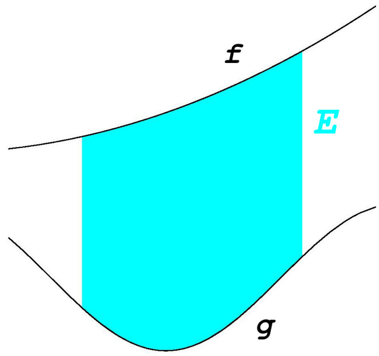
2.1 Area of a plane region

Area between two curves. Let functions f, g be continuous on an interval $\langle \alpha, \beta \rangle$ and let, for all $x \in \langle \alpha, \beta \rangle$, $g(x) \leq f(x)$. The area of the region

$$E = \{ [x, y] \in \mathbb{R}^2 ; \alpha \leq x \leq \beta, g(x) \leq y \leq f(x) \},$$

which we call an *elementary region*, is

$$P(E) = \int_{\alpha}^{\beta} (f(x) - g(x)) \, dx.$$



For computing area of a plane region which is a union of elementary regions we use:

Additivity. If a plane region A is a *union* of two non-overlapping regions B, C (that is, A and B has no joint interior points), then

$$P(A) = P(B) + P(C).$$

Identity. If A, B are two *identical* plane regions (for example, A is obtained from B by translation) then

$$P(A) = P(B).$$