9 Systems of linear equations

(Applied Mathematics — FAPPZ)

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1 Rank of a linear span

Definition 1. Let at least one of vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ is nonzero and let $\langle \mathbf{u}_1, \ldots, \mathbf{u}_k \rangle$ denote their linear span. *Rank of the linear span* $\langle \mathbf{u}_1, \ldots, \mathbf{u}_k \rangle$ we define as the *number* of its linearly independent generators. We denote it by $\dim\langle \mathbf{u}_1, \ldots, \mathbf{u}_k \rangle$. For the linear span of zero vector we put $\dim\langle \mathbf{o} \rangle = 0$.

In particular, if the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are linearly independent, then the rank $\dim\langle \mathbf{u}_1, \ldots, \mathbf{u}_k \rangle = k$.

Theorem 2. *Let $\boldsymbol{A}$ be a matrix whose rows are vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$. Then*

- *rank of their linear span $\langle \mathbf{u}_1, \ldots, \mathbf{u}_k \rangle$ equals to rank of the matrix $\boldsymbol{A}$;*
- *vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are linearly independent if and only if $\mathrm{h}(\boldsymbol{A}) = k$.*

2 Systems of linear exuations

2.1 Basic definitions

Definition 3. A system of linear equation

$$\begin{array}{ccccccc} a_{11}\,x_1 & + & a_{12}\,x_2 & +\ldots+ & a_{1n}\,x_n & = & b_1 \\ a_{21}\,x_1 & + & a_{22}\,x_2 & +\ldots+ & a_{2n}\,x_n & = & b_2 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{m1}\,x_1 & + & a_{m2}\,x_2 & +\ldots+ & a_{mn}\,x_n & = & b_m \end{array}, \tag{1}$$

where a_{ij}, b_i, $i = 1, \ldots, m$, $j = 1, \ldots, n$, are given real numbers, we call a *homogeneous system* if $b_1 = \cdots = b_m = 0$. If at least one of numbers $b_1, \ldots, b_m$ is nonzero then we call this system *nonhomogeneous.*

A vector $\mathbf{x} \in \mathbf{V}_n$, $\mathbf{x} = (x_1, \ldots, x_n)$ with $x_1, \ldots, x_n$ satisfying all the equations of the system we call a *solution* of the system.

Definition 4. The matrix

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

is called the *matrix of the system* (1). The matrix

$$\big(\boldsymbol{A}|\mathbf{b}\big) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array}\right)$$

(whose last column is a vector of right hand sides of system (1)) is called the *augmented matrix* of the system (1).

2.2 Homogeneous systems

$$\begin{array}{ccccccccc} a_{11}\,x_1 & + & a_{12}\,x_2 & + \ldots + & a_{1n}\,x_n & = & 0 \\ a_{21}\,x_1 & + & a_{22}\,x_2 & + \ldots + & a_{2n}\,x_n & = & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{m1}\,x_1 & + & a_{m2}\,x_2 & + \ldots + & a_{mn}\,x_n & = & 0 \end{array} \tag{2}$$

A homogeneous system (2) satisfies:

- The zero vector $\mathbf{x} = (x_1, \ldots, x_n) = (0, \ldots, 0)$ is always a solution of the homogeneous system.
- If $\mathrm{h}(\boldsymbol{A}) = n$, that is, the rank of the matrix of the system is equal to the number of variables, then the zero vector is *the only* solution of the homogeneous system.
- If $\mathrm{h}(\boldsymbol{A}) < n$, that is, the rank of the matrix of the system is less than the number of variables, then the linear span generated by all vectors of its solutions $\mathbf{x}$ has the rank $n - \mathrm{h}(\boldsymbol{A})$.

2.3 Nonhomogeneous systems

Theorem 5 (Frobenius)**.** *Nonhomogneous system of linear equations* (1) *has solution if and only if* $\mathrm{h}(\boldsymbol{A}) = \mathrm{h}\big((\boldsymbol{A}|\mathbf{b})\big)$, *that is the rank of the matrix of the system is equal to the rank of the augmented matrix of this system.*

How to express all solutions.
If the system (1) has a solution, then all its solutions can be expressed in the form $\mathbf{x} = \mathbf{u} + \mathbf{v}$, where $\mathbf{u}$ is one fixed (arbitrary) solution of (1) and $\mathbf{v}$ is any solution of the corresponding homogeneous system (2).