LINE IN THE PLANE, TANGENT LINE AND NORMAL LINE TO THE GRAPH OF A FUNCTION

1. Line in the plane

Consider the plane $\mathbb{R}^2$ with the Cartesian coordinate system. A point A in uniquely determined by an ordered pair of real numbers (coordinates) x, y. We write $A = [x, y]$.

Let p be a given line in the plane.

General equation of a line. Points $[x, y]$ of the line p satisfy the equation

$$p : \quad ax + by + c = 0,$$

where $a, b, c \in \mathbb{R}$ are constants such that at least one of the numbers a, b is *nonzero*.

- If $a = 0$, then the line p is parallel to the coordinate axis x.
- If $b = 0$, then the line p is parallel to the coordinate axis y.
- To find an equation of p (e. g. to determine the constants a, b, c) passing through given points $A = [x_1, y_1]$, $B = [x_2, y_2]$ we should solve the system of equations, which we obtain after substituting A, B into the general equation of p.
- The lines $p_1 : a_1x + b_1y + c_1 = 0$, $p_2 : a_2x + b_2y + c_2 = 0$ are *parallel* if and only if there exists a real number m, such that $a_2 = ma_1$ and $b_2 = mb_1$.
- The lines $p_1 : a_1x + b_1y + c_1 = 0$, $p_2 : a_2x + b_2y + c_2 = 0$ are *perpendicular* if and only if $a_1a_2 + b_1b_2 = 0$.

Slope equation of a line. If $b \neq 0$, that is the line p *is not parallel* with the y-axis, then its general equation can be modified to the form

$$p : \quad y = kx + q$$

($k = -\frac{a}{b}$, $q = -\frac{c}{b}$). The number k is called the *slope* of the line p.

- A line parallel with the y-axis *cannot* be expressed by slope equation.
- If $k = 0$ then the line p is parallel with the x-axis.
- A slope equation of p joining given points $A = [x_1, y_1]$, $B = [x_2, y_2]$, $x_1 \neq x_2$, we obtain, after simple calculation, from

  $$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \quad \text{or} \quad y - y_2 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_2).$$

  Consequently, the slope of this line p equals $k = \frac{y_2 - y_1}{x_2 - x_1}$.
- The lines $p_1 : \ y = k_1x + q_1$, $p_2 : \ y = k_2x + q_2$ are *parallel* if and only if their slopes are equal, that is, $k_1 = k_2$.
- Two nonvertical lines $p_1 : \ y = k_1x + q_1$, $p_2 : \ y = k_2x + q_2$ (that means that none of them is not parallel with coordinate axes) are *perpendicular* if and only if their slopes k_1 and k_2 satisfy $k_1k_2 = -1$.

2. Equations of a tangent line and of a normal to the graph of a function

Tangent. If $f'(a)$ denotes the first derivative of a function f in the point a then the line t,

$$t : y - f(a) = f'(a)(x - a),$$

is called the *tangent* to the graph of the function f at point $T = [a, f(a)]$.

Normal. (Normal is perpendicular to tangent.) The line n determined by the equation

$$n:\ x = a \quad \text{if } f'(a) = 0 \quad (n \text{ is parallel with } y\text{-axis}),$$

$$n:\ y - f(a) = \frac{-1}{f'(a)}(x - a) \quad \text{if } f'(a) \neq 0,$$

we call the *normal* to the graph of the function f at point $T = [a, f(a)]$.