IMPORTANT FORMULAE

1. **Inequalities.** For all real numbers $a, b, c \in \mathbb{R}$,

$$
\begin{aligned}
a < b \ \wedge\ b < c \ &\Longrightarrow\ a < c \\
a < b \ &\Longleftrightarrow\ a + c < b + c, \\
\text{for } c > 0, \quad a < b \ &\Longleftrightarrow\ c\,a < c\,b, \\
\text{for } c < 0, \quad a < b \ &\Longleftrightarrow\ c\,a > c\,b \quad (\text{especially: } a < b \iff -a > -b), \\
\text{for } a, b \geq 0 \text{ we have} \quad a < b \ &\Longleftrightarrow\ a^2 < b^2
\end{aligned}
$$

(analogous relations hold also for $\leq$, $>$ a $\geq$).

Important properties for solving inequalities:

$$
\begin{aligned}
ab > 0 \quad &\Longleftrightarrow \quad \text{either } a > 0 \ \wedge\ b > 0, \text{ or } a < 0 \ \wedge\ b < 0, \\
ab < 0 \quad &\Longleftrightarrow \quad \text{either } a > 0 \ \wedge\ b < 0, \text{ or } a < 0 \ \wedge\ b > 0.
\end{aligned}
$$

In particular,

$$
\begin{aligned}
a^2 > 0 \quad & \text{for every } a \in \mathbb{R} \setminus \{0\}, \\
a^2 = 0 \quad & \Longleftrightarrow \quad a = 0, \\
a^2 \geq 0 \quad & \text{for every } a \in \mathbb{R}.
\end{aligned}
$$

2. **Formulae with the second powers.**

$$
(A+B)^2 = A^2 + 2AB + B^2, \qquad (A-B)^2 = A^2 - 2AB + B^2,
$$
$$
A^2 - B^2 = (A+B)(A-B) \qquad (\text{where } A, B \in \mathbb{R}).
$$

3. **Square root in $\mathbb{R}$.** Square root $\sqrt{x}$ of a real number x is a *nonegative* real number a, which satisfies $a^2 = x$, that is,

$$
\sqrt{x} = a \quad \Longleftrightarrow \quad a \geq 0 \wedge a^2 = x.
$$

The fact $a^2 \geq 0$ for every $a \in \mathbb{R}$ implies that $\sqrt{x}$ makes sense in $\mathbb{R}$ only if $x \geq 0$.

4. **Absolute value.** It is given by $|x| = \sqrt{x^2}$, $x \in \mathbb{R}$. The following hold

$$
|-a| = |a|, \quad |ab| = |a|\,|b|, \quad \left|\frac{a}{b}\right| = \frac{|a|}{|b|} \quad |a+b| \leq |a| + |b|
$$

for $a, b \in \mathbb{R}$, whenever all the expressions are defined.

5. **Quadratic equation.** Consider the equation

$$
ax^2 + bx + c = 0 \qquad (\text{where } a, b, c \in \mathbb{R},\ a \neq 0)
$$

with variable x.
Number $D = b^2 - 4ac$ is called *diskriminant*. *Zero points* x_1, x_2 we compute by the formula:

$$
\begin{aligned}
D > 0 \quad &\Rightarrow \quad x_{1,2} = \tfrac{-b \pm \sqrt{D}}{2a} && \text{(2 different real zero point)}, \\
D = 0 \quad &\Rightarrow \quad x_1 = x_2 = -\tfrac{b}{2a} && \text{(1 double real zero point)}, \\
D < 0 \quad &\Rightarrow \quad x_{1,2} = \tfrac{-b \pm \mathrm{i}\sqrt{|D|}}{2a} \quad (\text{kde } \mathrm{i}^2 = -1) && \text{(2 complex zero points)}.
\end{aligned}
$$

Observe that, $ax^2 + bx + c = a(x - x_1)(x - x_2)$. If $a = 1$, then $x^2 + bx + c = (x - x_1)(x - x_2) = x^2 - (x_1 + x_2)x + x_1 x_2$, and so $b = -(x_1 + x_2)$, $c = x_1 x_2$.

6. **Binomial theorem.** For $A, B \in \mathbb{R},\ n \in \mathbb{N}$,

$$(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^{n-k} B^k = \binom{n}{0} A^n + \binom{n}{1} A^{n-1} B + \cdots + \binom{n}{n-1} A B^{n-1} + \binom{n}{n} B^n,$$

where $\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$, $k! = k \cdot (k-1) \cdot \dots \cdot 2 \cdot 1$.

7. **Powers and radicals.** If $m, n \in \mathbb{N}$, $r, s \in \mathbb{R}$, then, for all $x, y \in \mathbb{R}$, for which both sides make sense,

$$x^r\, x^s = x^{r+s}, \quad x^r : x^s = \frac{x^r}{x^s} = x^{r-s}, \quad x^{-r} = \frac{1}{x^r}, \quad x^0 = 1,$$

$$x^r\, y^r = (x\, y)^r, \quad \frac{x^r}{y^r} = \left(\frac{x}{y}\right)^r, \quad \frac{1}{x^r} = \left(\frac{1}{x}\right)^r, \quad (x^r)^s = x^{rs},$$

$$\sqrt[n]{x} = x^{\frac{1}{n}}, \quad \sqrt[n]{x\, y} = \sqrt[n]{x}\,\sqrt[n]{y}, \quad \sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}},$$

$$\sqrt[n]{x^r} = \left(\sqrt[n]{x}\right)^r = x^{\frac{r}{n}}, \quad \sqrt[n]{\sqrt[m]{x}} = \sqrt[m]{\sqrt[n]{x}} = \sqrt[n \cdot m]{x}, \quad \sqrt[n]{x^n} = 1$$

8. **Exponential and logarithmic function.** Assume that $a \in (0,1) \cup (1, \infty)$. Then

$$\begin{array}{llll} a^y = x & \Leftrightarrow \quad y = \log_a x & (\text{for } x > 0,\ y \in \mathbb{R}) & (\text{logarithm}), \\ 10^y = x & \Leftrightarrow \quad y = \log x & (\text{for } x > 0,\ y \in \mathbb{R}) & (\text{decadic logarithm}), \\ e^y = x & \Leftrightarrow \quad y = \ln x & (\text{for } x > 0,\ y \in \mathbb{R}) & (\text{natural logarithm}) \end{array}$$

($e = 2{,}718\ldots$ is the *Euler number*).
For $u, v > 0$, $s \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\log_a (u\, v) = \log_a u + \log_a v, \quad \log_a \left(\frac{u}{v}\right) = \log_a u - \log_a v,$$

$$\log_a \left(u^s\right) = s\, \log_a u, \quad \log_a \left(\sqrt[n]{u}\right) = \frac{1}{n} \log_a u,$$

$$\log_a 1 = 0, \quad \log_a a = 1,$$

$$s = \log_a \left(a^s\right), \quad a^{\log_a u} = u.$$

9. **Trigonometric functions.**

Formulae for trigonometric functions. If $k \in \mathbb{Z}$, then for all $x \in \mathbb{R}$, for which both sides make sense,

$$\cos x = \sin\left(\frac{\pi}{2} - x\right), \quad \sin^2 x + \cos^2 x = 1, \quad \operatorname{tg} x = \frac{\sin x}{\cos x}, \quad \operatorname{cotg} x = \frac{\cos x}{\sin x},$$

$$\sin(-x) = -\sin x, \quad \cos(-x) = \cos x, \quad \operatorname{tg}(-x) = -\operatorname{tg} x, \quad \operatorname{cotg}(-x) = -\operatorname{cotg} x,$$

$$\sin(x \pm 2k\pi) = \sin x, \quad \cos(x \pm 2k\pi) = \cos x, \quad \operatorname{tg}(x \pm k\pi) = \operatorname{tg} x, \quad \operatorname{cotg}(x \pm k\pi) = \operatorname{cotg} x.$$

For all $x, y \in \mathbb{R}$, for which both sides make sense,

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y, \quad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$

$$\operatorname{tg}(x \pm y) = \frac{\operatorname{tg} x \pm \operatorname{tg} y}{1 \mp \operatorname{tg} x \operatorname{tg} y}, \quad \operatorname{cotg}(x \pm y) = \frac{\operatorname{cotg} x \operatorname{cotg} y \mp 1}{\operatorname{cotg} y \pm \operatorname{cotg} x},$$

$$\sin 2x = 2 \sin x \cos x, \quad \cos 2x = \cos^2 x - \sin^2 x,$$

$$\operatorname{tg} 2x = \frac{2 \operatorname{tg} x}{1 - \operatorname{tg}^2 x}, \quad \operatorname{cotg} 2x = \frac{\operatorname{cotg}^2 x - 1}{2 \operatorname{cotg} x},$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}.$$

Table of important values of trigonometric functions.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\operatorname{tg} x$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	–	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	–	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$
$\operatorname{cotg} x$	–	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	$-\frac{\sqrt{3}}{3}$	-1	$-\sqrt{3}$	–	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	$-\frac{\sqrt{3}}{3}$	-1	$-\sqrt{3}$

10. **Circular functions.**

$y = \arcsin x$	$\Longleftrightarrow$	$x = \sin y$	for	$y \in \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$, $x \in \langle -1, 1 \rangle$	(arcsin =arcsine),
$y = \arccos x$	$\Longleftrightarrow$	$x = \cos y$	for	$y \in \langle 0, \pi \rangle$, $x \in \langle -1, 1 \rangle$	(arccos =arccosine),
$y = \operatorname{arctg} x$	$\Longleftrightarrow$	$x = \operatorname{tg} y$	for	$y \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $x \in \mathbb{R}$	(arctg =arctangent),
$y = \operatorname{arccotg} x$	$\Longleftrightarrow$	$x = \operatorname{cotg} y$	for	$y \in (0, \pi)$, $x \in \mathbb{R}$	(arccotg =arccotangent).

11. **Domain of a real function of one real variable.** When determining a domain $\mathcal{D}(f)$ of all x for which the function formula $f(x)$ makes sense we should start with the conditions.

The most important conditions are listed in the following table.

expression at the denominator must be nonzero:	$\frac{1}{g(x)}$ makes sense only if $g(x) \neq 0$
argument of square root must be nonegative:	$\sqrt{g(x)}$ makes sense only if $g(x) \geq 0$
argument of even root must be nonegative:	$\sqrt[2k]{g(x)}$ makes sense only if $g(x) \geq 0$
argument of log *must be positive*:	$\log\big(g(x)\big)$ makes sense only if $g(x) > 0$
argument of arcsin *must belong to* $\langle -1, 1 \rangle$:	$\arcsin\big(g(x)\big)$ makes sense only if $-1 \leq g(x) \leq 1$
argument of arccos *must belong to* $\langle -1, 1 \rangle$:	$\arccos\big(g(x)\big)$ makes sense only if $-1 \leq g(x) \leq 1$