2. DERIVOVÁNÍ FUNKCE DVOU PROMĚNNÝCH

CVIČENÍ Z MATEMATIKY 3 (DOPORUČENÉ ÚLOHY)

1. Výpočet parciálních derivací podle vzorců a pravidel

Pro danou funkci dvou proměnných $f : z = f(x, y)$ vypočítejte obě parciální derivace 1. řádu, tj. $\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x}(x, y)$ a $\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y}(x, y)$ (podle vzorců a pravidel).

Základní úlohy.

1) $z = x^3 + 3x^2y + y^2$ 2) $z = x\,\ln(x^2 + y^2)$ 3) $z = y^2 \operatorname{arctg} \frac{x}{y}$

4) $z = (\sin x)^{\cos y}$ 5) $z = x^4 \cos^2 y$ 6) $z = \dfrac{x}{y} - x \sin y$

Zkouškové úlohy.

7) $z = \operatorname{arctg} \frac{x}{y} + \ln \sqrt{\frac{x-y}{x+y}}$ 8) $z = \ln \frac{\sqrt{a+ye^x}-\sqrt{a}}{\sqrt{a+ye^x}+\sqrt{a}}$ 9) $z = x^2y \operatorname{arctg} \sqrt{xy} - \operatorname{arctg}(-1)$

10) $z = \ln(\sqrt{xy} + x^3y) - x^2 \ln y$ 11) $z = \ln(x + \sqrt{x^2 + y^2})$ 12) $z = \arcsin \sqrt{\frac{x^2-y^2}{x^2+y^2}}$

Obtížnější úlohy. Pro danou funkci dvou proměnných $f : z = f(x, y)$ vypočítejte všechny parciální derivace 2. řádu, tj. $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2}(x, y)$ a $\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial y^2}(x, y)$ a $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y}(x, y)$.

13) $z = y \ln x$ 14) $z = \operatorname{arctg} \frac{x-y}{x+y}$ 15) $z = \sin(x^2 + y^3)$

Výsledky.

1) $\dfrac{\partial z}{\partial x} = 3x^2 + 6xy, \quad \dfrac{\partial z}{\partial y} = 3x^2 + 2y$

2) $\dfrac{\partial z}{\partial x} = \ln(x^2 + y^2) + \dfrac{2x^2}{x^2 + y^2}, \quad \dfrac{\partial z}{\partial y} = \dfrac{2xy}{x^2 + y^2}$

3) $\dfrac{\partial z}{\partial x} = \dfrac{y^3}{x^2 + y^2}, \quad \dfrac{\partial z}{\partial y} = 2y \operatorname{arctg} \frac{x}{y} - \dfrac{xy^2}{x^2 + y^2}$

4) $\dfrac{\partial z}{\partial x} = \cos y (\sin x)^{\cos y} \operatorname{cotg} x, \quad \dfrac{\partial z}{\partial y} = -(\sin x)^{\cos y} \sin y \, \ln \sin x$

5) $\dfrac{\partial z}{\partial x} = 4x^3 \cos^2 y, \quad \dfrac{\partial z}{\partial y} = -x^4 \sin 2y$

6) $\dfrac{\partial z}{\partial x} = \dfrac{1}{y} - \sin y, \quad \dfrac{\partial z}{\partial y} = -\dfrac{x}{y^2} - x \cos y$

7) $\dfrac{\partial z}{\partial x} = \dfrac{2x^2y}{x^4 - y^4}, \quad \dfrac{\partial z}{\partial y} = \dfrac{-2x^3}{x^4 - y^4}$

8) $\dfrac{\partial z}{\partial x} = \sqrt{\dfrac{a}{a + ye^x}}, \quad \dfrac{\partial z}{\partial y} = \dfrac{1}{y} \sqrt{\dfrac{a}{a + ye^x}}$

9) $\dfrac{\partial z}{\partial x} = 2xy \operatorname{arctg} \sqrt{xy} + \dfrac{x^2y^2}{2\sqrt{xy}(1 + xy)}, \quad \dfrac{\partial z}{\partial y} = x^2 \operatorname{arctg} \sqrt{xy} + \dfrac{x^3y}{2\sqrt{xy}(1 + xy)}$

10) $\dfrac{\partial z}{\partial x} = \dfrac{1}{\sqrt{xy} + x^3y} \Big(\dfrac{y}{2\sqrt{xy}} + 3x^2y \Big) - 2x \ln y, \quad \dfrac{\partial z}{\partial y} = \dfrac{1}{\sqrt{xy} + x^3y} \Big(\dfrac{x}{2\sqrt{xy}} + x^3 \Big) - \dfrac{x^2}{y}$

11) $\dfrac{\partial z}{\partial x} = \dfrac{1}{\sqrt{x^2 + y^2}}, \quad \dfrac{\partial z}{\partial y} = \dfrac{y}{x\sqrt{x^2 + y^2} + x^2 + y^2}$

12) $\dfrac{\partial z}{\partial x} = \dfrac{xy^2\sqrt{2x^2 - 2y^2}}{|y|\,(x^4 - y^4)}, \quad \dfrac{\partial z}{\partial y} = \dfrac{-x^2y\sqrt{2x^2 - 2y^2}}{|y|\,(x^4 - y^4)}$

13) $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{-y}{x^2}$, $\dfrac{\partial^2 z}{\partial y^2} = 0$ $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{1}{x}$

14) $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{-2xy}{(x^2+y^2)^2}$, $\dfrac{\partial^2 z}{\partial y^2} = \dfrac{2xy}{(x^2+y^2)^2}$ $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{x^2-y^2}{(x^2+y^2)^2}$

15) $\dfrac{\partial^2 z}{\partial x^2} = 2\cos(x^2+y^3) - 4x^2\sin(x^2+y^3)$,

$\dfrac{\partial^2 z}{\partial y^2} = 6y\cos(x^2+y^3) - 9y^4\sin(x^2+y^3)$,

$\dfrac{\partial^2 z}{\partial x \partial y} = -6xy^2\sin(x^2+y^3)$

2. Výpočet parciálních derivací v bodě dosazením

Základní úlohy. 1) Vypočítejte parciální derivace $\frac{\partial f}{\partial x}(x,y)$ a $\frac{\partial f}{\partial y}(x,y)$ funkce f a hodnotu $\frac{\partial f}{\partial x}(0,1)$, je-li

$$f(x,y) = \sqrt{\operatorname{arctg}(y-x)} + 3x^2y + \sqrt{\pi}.$$

2) Vypočítejte parciální derivace $\frac{\partial f}{\partial x}(x,y)$ a $\frac{\partial f}{\partial y}(x,y)$ funkce f a hodnotu $\frac{\partial f}{\partial y}(\pi,\pi^2)$, je-li

$$f(x,y) = \ln(x^2 + 3x\sqrt{y}) - y\sin x.$$

3) Vypočítejte parciální derivaci $\frac{\partial^2 f}{\partial y^2}(x,y)$ funkce f a hodnotu $\frac{\partial^2 f}{\partial y^2}(1,2)$, je-li

$$f(x,y) = \operatorname{arctg}(y-x) + \arcsin(\tfrac{1}{2}).$$

4) Vypočítejte parciální derivaci $\frac{\partial^2 f}{\partial x \partial y}(x,y)$ funkce f a hodnotu $\frac{\partial^2 f}{\partial x \partial y}(1,2)$, je-li

$$f(x,y) = \cos(2x - xy) + \cos\tfrac{\pi}{4}.$$

Výsledky. 1)

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{2\sqrt{\operatorname{arctg}(y-x)}} \cdot \frac{1}{1+(y-x)^2} \cdot (-1) + 6xy,$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{1}{2\sqrt{\operatorname{arctg}(y-x)}} \cdot \frac{1}{1+(y-x)^2} \cdot 1 + 3x^2, \qquad \frac{\partial f}{\partial x}(0,1) = \frac{1}{2\sqrt{\pi}}$$

2)

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{x^2+3x\sqrt{y}} \cdot (2x + 3\sqrt{y}) - y\cos x,$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{1}{x^2+3x\sqrt{y}} \cdot \frac{3x}{2\sqrt{y}} - \sin x \qquad \frac{\partial f}{\partial y}(\pi,\pi^2) = \frac{3}{8\pi^2}$$

3)

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{2x-2y}{(1+(y-x)^2)^2}, \qquad \frac{\partial^2 f}{\partial y^2}(1,2) = -\frac{1}{2}$$

4)

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \sin(2x-xy) + x(2-y)\cos(2x-xy), \qquad \frac{\partial^2 f}{\partial x \partial y}(1,2) = 0$$